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THE PLANE PROBLEM OF HYDROELASTIC STABILITY FOR A HINGE-SUPPORTED PLATE*

B.YA. KANTOR and M.P. KVITNITSKII

The plane problem of hydroelasticity on the stability of a hinge-supported plate of infinite span placed in a rigid screen is considered in the case of unilateral flow of an ideal incompressible fluid. An analytical representation is obtained for the matrix elements of the averaged aerodynamic loads. The possibility of using the method reduction in the problem under consideration, i.e., of replacing the infinite determinant by a truncated determinant is investigated. Relations are obtained for the flutter velocity as a function of the hydroelasticity and axial force parameters. The problem under consideration was solved in /1-3/ by different methods, where, by assuming the convergence of the infinite determinant to which application of the Bubnov-Galerkin method leads, consideration was confined to two coordinate functions and the forces acting on the fluid side were determined numerically. Only the boundary of the static stability domain was found.

1. Formulation of the hydroelasticity problem. We will write the equation of the cylindrical vibrations of a plate extended in the stream direction by forces H as follows:

$$Dw_{xxxx} - Hw_{xx} + \varepsilon h\rho_0 w_t + h\rho_0 w_{tt} = p \tag{1.1}$$

Here $w(x, t)$ and $p(x, t)$ are the plate deflection and the fluid pressure thereon, D is the bending stiffness, ε is the damping coefficient, h is the thickness, and ρ_0 is the specific density of the plate material.

The hinge clamping boundary conditions at the points $x = \pm a$ have the form

$$w = w_{xx} = 0 \tag{1.2}$$

The potential of the perturbed fluid velocities $\Phi(x, z, t)$ satisfies the Laplace equation, the damping condition, and the non-penetration condition

$$\Phi_{xx} + \Phi_{zz} = 0, \quad z \leqslant 0 \tag{1.3}$$

$$\lim_{r_*\to\infty} \nabla\Phi = 0, \quad r_* = \sqrt{x^2 + z^2} \tag{1.4}$$

$$\Phi_z = w_t + Vw_x, \quad x \in [-a; a], \quad z = 0 \tag{1.5}$$
$$\Phi_z = 0, \quad x \notin [-a; a], \quad z = 0$$

Here V is the velocity of unperturbed fluid motion.

Using the representation of a harmonic function in the form of the potential difference of a simple and double layer (for instance /4/), and taking into account that the cosine of the angle between the tangent plane and the normal to the surface $z = w(x, t)$ is small compared with unity, we obtain by virtue of (1.5)

$$\Phi(x, 0, t) = 1/\pi \int_{-a}^{a} (w_t + Vw_x)_{x=x'} \ln \frac{a}{|x - x'|} dx' \tag{1.6}$$

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The fluid pressure is determined from the linearized Cauchy-Lagrange integral (the static pressure is taken to be equal to zero, and ρ is the fluid density)

$$p = -\rho\,(\Phi_t + V\Phi_x)_{z=0} \tag{1.7}$$

Substituting (1.6) and (1.7) into (1.1), we obtain the following integrodifferential equation for the flow around the plate:

$$Dw_{xxxx} - Hw_{xx} + \varepsilon h\rho_0 w_t + h\rho_0 w_{tt} = \frac{\rho}{\pi}\int_{-a}^{a}(w_{tt})_{x=x'}\ln\frac{|x-x'|}{a}\,dx' + \tag{1.8}$$

$$\frac{2\rho V}{\pi}\int_{-a}^{a}(w_t)_{x=x'}\frac{dx'}{x-x'} + \frac{\rho V^2}{\pi}\int_{-a}^{a}(w_x)_{x=x'}\frac{dx'}{x-x'}$$

which agrees with that presented in /1/ for the problem of circulation-free flow around a symmetrical slender wing.

2. Solution of the hydrodynamic problem. We will find the solution of (1.8) by the Bubnov-Galerkin method. We take

$$w(x,t) = \sum_{k=1}^{\infty}\left(f_{2k-1}(t)\cos\frac{(2k-1)\pi x}{2a} + f_{2k}(t)\sin\frac{k\pi x}{a}\right) \tag{2.1}$$

We introduce the notation

$$\xi = \frac{x}{a},\quad \xi' = \frac{x'}{a},\quad N = \frac{H(2a)^2}{D\pi^2},\quad \Omega_0 = \frac{\pi^2}{(2a)^2}\sqrt{\frac{D}{\rho_0 h}} \tag{2.2}$$

$$\Omega_j = \Omega_0\sqrt{j^4 + Nj^2},\quad \omega_j = \frac{\Omega_j}{\Omega_0},\quad \tau = \Omega_0 t$$

$$g = \frac{\varepsilon}{\Omega_0},\quad u = \frac{V}{2a\Omega_0},\quad c = \frac{2\rho a}{\rho_0 h}$$

$$\varphi_1(\xi,\beta) = \int_{-1}^{1}\frac{\sin\beta\xi'}{\xi-\xi'}\,d\xi',\quad \varphi_2(\xi,\beta) = \int_{-1}^{1}\frac{\cos\beta\xi'}{\xi-\xi'}\,d\xi'$$

where the integrals φ_1 and φ_2 should be understood in the Cauchy principal value sense.

Substituting (2.1) into (1.8), applying the Bubnov-Galerkin procedure and taking account of (2.2), we obtain a system of ordinary differential equations in $f_j(\tau)$

$$f_j^{\cdot\cdot} + gf_j^{\cdot} + \omega_j^2 f_j + F_j(\tau) = 0 \quad (j = 1, 2, \ldots) \tag{2.3}$$

$$F_j = \sum_{k=1}^{\infty}\left({}^1/{}_2 c\Phi_{jk}f_k^{\cdot\cdot} + 2ucT_{jk}f_k^{\cdot} - 2u^2cL_{jk}f_k\right)$$

$$\Phi_{jk} = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\frac{j\pi\xi}{2}\cos\frac{k\pi\xi'}{2}\ln|\xi-\xi'|\,d\xi'\,d\xi, & j = 2m-1,\ k = 2n-1 \\ -\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\sin\frac{j\pi\xi}{2}\sin\frac{k\pi\xi'}{2}\ln|\xi-\xi'|\,d\xi'\,d\xi, & j = 2m,\ k = 2n \\ 0, & j+k - \text{odd} \end{cases}$$

$$T_{jk} = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\varphi_1\left(\xi,\frac{k\pi}{2}\right)\cos\frac{j\pi\xi}{2}\,d\xi, & j = 2m-1,\quad k = 2n \\ -\dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\varphi_2\left(\xi,\frac{k\pi}{2}\right)\sin\frac{j\pi\xi}{2}\,d\xi, & j = 2m,\quad k = 2n-1 \\ 0, & j+k - \text{even} \end{cases}$$

$$L_{jk} = \begin{cases} \dfrac{k}{2}\displaystyle\int_{-1}^{1}\varphi_1\left(\xi,\frac{k\pi}{2}\right)\cos\frac{j\pi\xi}{2}\,d\xi, & j = 2m-1,\quad k = 2n-1 \\ -\dfrac{k}{2}\displaystyle\int_{-1}^{1}\varphi_2\left(\xi,\frac{k\pi}{2}\right)\sin\frac{j\pi\xi}{2}\,d\xi, & j = 2m,\quad k = 2n \\ 0, \quad j+k - \text{odd} & (n, m = 1, 2, \ldots) \end{cases}$$

The functions φ_1 and φ_2 can be represented in the form

$$\varphi_1(\xi, \beta) = A \sin \beta\xi - B \cos \beta\xi, \quad \varphi_2(\xi, \beta) = A \cos \beta\xi + B \sin \beta\xi \tag{2.4}$$
$$A = \mathrm{ci}(\beta + \beta\xi) - \mathrm{ci}(\beta - \beta\xi), \quad B = \pi + \mathrm{si}(\beta + \beta\xi) + \mathrm{si}(\beta - \beta\xi)$$
$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt, \quad \mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt$$

Substituting (2.4) into (2.3) and integrating, we obtain a representation of the non-zero elements of the matrices of the averaged aerodynamic loads in terms of known functions

$$L_{jk} = \left[\frac{\pi k}{2} + k\, \mathrm{si}(\pi k) + \frac{(-1)^k - 1}{\pi}\right]\delta_{jk} + \frac{2(-1)^{(j+k)/2}}{\pi(k^2 - j^2)}[\mathrm{ci}(\pi j) - \mathrm{ci}(\pi k) + \ln k - \ln j](1 - \delta_{jk}) \tag{2.5}$$
$$\Phi_{jk} = \frac{8}{\pi^3 k^2}\left[\mathrm{ci}(\pi k) - G - (-1)^k \ln 2 - \ln\frac{\pi k}{2} - \frac{\pi}{2} L_{kk}\right]\delta_{jk} + \frac{8(-1)^{(k+2jk+j)/2}}{\pi^3 kj(j^2 - k^2)}\Big[(k^2 - j^2)(G + (-1)^{kj}\ln 2) +$$
$$k^2\left(-\mathrm{ci}(\pi j) + \ln\frac{\pi j}{2}\right) + j^2\left(\mathrm{ci}(\pi k) - \ln\frac{\pi k}{2}\right)\Big](1 - \delta_{jk})$$
$$L_{jk} = \Phi_{jk} = 0, \quad j + k = 2n - 1$$
$$T_{jk} = (-1)^{(k+j-1)/2}\left\{\frac{\pi^2 j}{2(j^2 - k^2)} + \frac{j\, \mathrm{si}(\pi k) + k\, \mathrm{si}(\pi j)}{j^2 - k^2} \times [1 + \mathrm{sign}(j - k)] + \frac{j[\mathrm{si}(\pi j) - \mathrm{si}(\pi k)]}{j^2 - k^2}[1 - \mathrm{sign}(j - k)]\right\}$$
$$T_{jk} = -T_{kj}, \quad j = 2n - 1,\ k = 2m; \quad T_{jk} = 0,\ j + k = 2n$$
$$(n, m = 1, 2, \ldots)$$

(δ_{jk} is the Kronecker delta, and $G = 0.577\ldots$ is Euler's constant).

In the case under consideration the matrices L_{jk} and Φ_{jk} are symmetric and satisfy reciprocity relationships /5/ characteristic for non-conservative problems of the theory of elastic stability $T_{jk} = (-1)^{k+j} T_{kj}$.

The solution of system (2.3) will be sought in the form $f_j = A_j e^{s\tau}$.

An uncambered plate shape is stable if all the roots of the characteristic equation

$$\det\left[(s^2 + gs + \omega_j^2)\delta_{jk} + {}^1/_2 c\Phi_{jk}s^2 + 2ucT_{jk}s - 2u^2cL_{jk}\right] = 0 \tag{2.6}$$

lie in the left half-plane /5/. The least value of u for which at least one of the roots s will be in the right half-plane is the critical velocity parameter.

3. Investigation of the convergence. Using the asymptotic representations of the integral sine and cosine /6/

$$\mathrm{si}(x) \sim -\frac{\cos x}{x}, \quad \mathrm{ci}(x) \sim \frac{\sin x}{x}, \quad x \gg 1$$

we obtain an estimate of the absolute values of the matrix elements of the averaged aerodynamic loads for $j, k \gg 1$

$$\Phi_{jk} \leqslant A_1\left[\frac{\delta_{jk}}{k} + \frac{|k^2 - j^2| + k^2 \ln j + j^2 \ln k}{jk|k^2 - j^2|}(1 - \delta_{jk})\right]$$
$$T_{jk} \leqslant A_2\left[\frac{1}{|k - j|} + \frac{k^2 + j^2}{jk|k^2 - j^2|}\right], \quad A_3 B_{jk} \leqslant |L_{jk}| \leqslant A_4 B_{jk}$$
$$B_{jk} = k\delta_{jk} + jk\frac{|\ln k - \ln j|}{|k^2 - j^2|}(1 - \delta_{jk})$$

Hence and henceforth A_i are certain constants independent of k and j.

As we know /5/, an infinite determinant of the form

$$\Delta = |\delta_{jk} + d_{jk}| \tag{3.1}$$

converges if the double series

$$\sum_j \sum_k |d_{jk}| \tag{3.2}$$

converges (here and henceforth summation is over j and k between 1 and ∞).
The determinant satisfying this condition is called normal.

We will represent the determinant (2.6) in the form (3.1). To do this we divide the j-th row by ω_j and the k-th column by ω_k. We obtain, after reduction

$$d_{jk} = \frac{s^2 + gs}{\omega_k^2}\delta_{jk} + {}^1/_2 cs^2 \frac{\Phi_{jk}}{\omega_j\omega_k} + 2ucs\frac{T_{jk}}{\omega_j\omega_k} - 2u^2 c\frac{L_{jk}}{\omega_j\omega_k}$$

It follows from (2.2) for a membrane for $D = 0$ that $\omega_k = k\sqrt{N}$. Since $|k-j| \geqslant \sqrt{k} + \sqrt{j} \geqslant 2j^{1/4}k^{1/4}$ and $k^2 + j^2 < kj(j+k)$, we have

$$\frac{|\Phi_{jk}|}{\omega_j\omega_k} < |A_1\left[\frac{\delta_{jk}}{k^3} + \frac{1-\delta_{jk}}{j^2k^2} + \frac{1-\delta_{jk}}{j^{5/4}k^{5/4}}\right], \quad \frac{|T_{jk}|}{\omega_j\omega_k} < \frac{2A_2}{j^{5/4}k^{5/4}} \qquad (3.3)$$

Similar estimates are obviously not only conserved but also magnified for $H = 0$ and $\omega_k = k^2$ in the case of a plate. Thus, the convergence of series (3.2) depends on the convergence of the series

$$S = \sum_j \sum_k \frac{|L_{jk}|}{\omega_j\omega_k}$$

We will examine first the case of a plate; then

$$\frac{|L_{jk}|}{\omega_j\omega_k} < |A_4\left[\frac{\delta_{jk}}{k^3} + \frac{1-\delta_{jk}}{j^{5/4}k^{5/4}}\right] \qquad (3.4)$$

The double series comprised of the right-hand sides of inequalities (3.3) and (3.4) converge, from which the convergence of the series (3.2) follows. Therefore, the determinant (2.6) for a plate is in the class of convergent (normal) determinants.

Let us now consider the case of a membrane, then

$$S > A_3\left[\sum_j \frac{1}{j} + 2\sum_j \sum_{k<j} \frac{\ln j - \ln k}{j^2 - k^2}(1 - \delta_{jk})\right]$$

We take $j/k = n$, where $1 < n < j$ and we introduce into the considerations the function

$$f(n) = n^2 \ln n / [j^2(n^2 - 1)]$$

It can be shown that for $n \in [1; +\infty]$ the minimum of $f(n)$ is reached for $n = 1$ and equals $1/(2j^2)$. Taking into account that the number of non-zero terms in the k-th column equals either $j/2$ or $(j \pm 1)/2$ but not less than $j/6$, we obtain

$$S > A_5 \sum_j 1/j$$

i.e., as for a supersonic flow /5/, the sufficient criterion for series (3.2) to converge is not satisfied in the case of a membrane.

The convergence of the determinant (2.6) was investigated numerically with a different number (up to twelve) coordinate functions retained in the expansion of the deflection. Results of the calculations showed that the determinant (2.6) converges even in the case of a membrane but the contribution of the higher modes to the pattern of vibrations is considerably more substantial than for a plate.

For instance, for a plate the amplitudes of the third and fifth modes referred to the amplitude of the first mode are 0.44 and 0.23%, respectively, while for a membrane they are 8 and 1%. Consequently, if a two-term approximation is justified for a plate, in the case of a membrane it is best to use a greater number of coordinate functions.

In the general case of a plate where D and H differ from zero, $\omega_k^2 = k^4 + Nk^2$. A finite number k can obviously always be selected such that the inequality $\omega_k > A_6 k^{11/8}$ is always satisfied. Then we have for sufficiently large j and k

$$\frac{|L_{jk}|}{\omega_j\omega_k} < A_7\left[\frac{\delta_{jk}}{k^{22/8}} + \frac{|1-\delta_{jk}}{j^{9/8}k^{9/8}}\right]$$

and, therefore, the series (3.2) converges, though the convergence will be slower the greater the value of N, i.e., the more the plate approximates to a membrane.

Thus for subsonic flow the determinant (2.6) converges for both a plate and a membrane.

4. **Numerical results.** Retaining the first two terms in expansion (2.1), we obtain that flutter precedes static buckling (divergence in the first mode). An analogous pattern

was also observed in experiments /7/. The critical velocity of the divergence for a plate equals

$$V_* = 1.137\left[\frac{D\pi^3}{\rho(2a)^3}\right]^{1/2} \tag{4.1}$$

The value 1.440 is obtained in /1-3/ for the coefficient on the right-hand side of (4.1) Utilization of a greater number of terms in expansion (2.1) does not alter the result in practice (for instance, keeping four terms of the series reduces the coefficient in question by just 0.04%).

When taking account of two terms of the expansion (2.1), the condition for the generation of flutter has the form

$$\Delta_{12} + \Delta_{21} + T_{12}{}^2c^2u^2 - 4\Delta_{12}\Delta_{21} = 0$$
$$\Delta_{jk} = (1 + {}^1/_2 c\Phi_{jj})(\omega_k{}^2 - 2cu^2L_{kk}),\quad j, k = 1, 2 \tag{4.2}$$

The value of g was assumed to be zero in deriving relationship (4.2), since it has been shown /2/ that damping exerts no influence on the magnitude of the flutter velocity.

The results of computing the divergence and flutter velocities for $N = 0$ are shown in Fig.1 (curves *1* and *2* correspond to (4.1) and (4.2)). The boundary obtained for the static stability domain (curve *1*) agrees with that presented in /1-3/. The solution of the problem of determining the flutter velocity was not obtained in the papers mentioned. Curves *3* and *4*, outlining the domain within which the desired solution should be found are constructed from the results of a qualitative analysis performed in /2/. The boundary of the flutter domain is obtained in this paper when the first two terms of the series are kept in the expansion (2.1) (curve *2*).

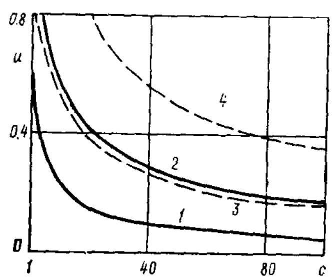

Fig.1

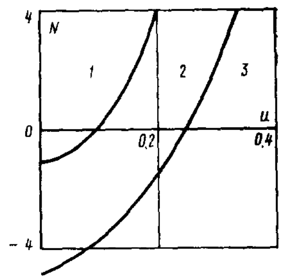

Fig.2

The boundaries of the instability domains are presented in Fig.2 as functions of the velocity and axial force for a plate with parameter $c = 50$. (*1* is the stability domain, *2* is divergence, while *3* is the flutter). The subsonic flow, unlike the supersonic flow (/8/, for example exerts a destabilizing influence on the plate by reducing the magnitude of the critical compressive force. For compressive forces significantly exceeding the critical value in a vacuum ($N = -1$) the plate at once drops into the flutter domain. An analogous pattern holds for supersonic flow also.

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